ELECTRICAL CONDUCTIVITY OF A PARTIALLY IONIZED GAS MIXTURE IN A MAGNETIC FIELD

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A considerable number of papers [1–5] has been devoted to determining the electrical conductivity of a partially ionized gas on the basis of kinetic theory. In so doing, a three-component plasma model (electrons, ions, neutrals) is generally employed. The general expressions for the electrical conductivity of a many-component system are fairly complicated [1], and the calculation of their determinants is most laborious.

The case of a N-component gas mixture in which one of the components is partially ionized (N + 2-component plasma) is considered below. A series of simplifications in the solution of the initial system of equations allows one to represent the expressions for the electrical conductivity of such a mixture in the same form as for the three-component plasma case, but with certain effective parameter values. The results obtained correspond to the "second approximation" of Cowling [1, 6].

As our initial system of equations we use the transport equations for diffusion velocities $\mathbf{w}_\gamma = \mathbf{u}_\gamma - \mathbf{u}$ and for the reduced relative heat fluxes $\mathbf{r}_\gamma = \mathbf{h}_\gamma/p_\gamma$ given in [4]. Omitting terms with pressure and temperature gradients and neglecting "viscous" transfer of momentum and the temperature difference of components (a similar system was employed in [3] for a three-component plasma), we have

$$\sum_\beta \lambda_{\alpha\beta}\left[\mathbf{w}_\alpha - \mathbf{w}_\beta + \frac{m_\beta}{m_\alpha + m_\beta}\zeta_{\alpha\beta}\left(\mathbf{r}_\alpha - \frac{m_\alpha}{m_\beta}\mathbf{r}_\beta\right)\right] = n_\alpha e_\alpha \mathbf{E}' + $$
$$+ n_\alpha e_\alpha(\mathbf{w}_\alpha \times \mathbf{H}) - \frac{\rho_\alpha}{\rho}(\mathbf{j}\times\mathbf{H}) \qquad (\mathbf{E}' = \mathbf{E} + \mathbf{u}\times\mathbf{H}), \quad (1)$$
$$\sum_\beta \lambda_{\alpha\beta}\left[b_{\alpha\beta}\mathbf{r}_\alpha + b'_{\alpha\beta}\mathbf{r}_\beta + \frac{m_\beta}{m_\alpha + m_\beta}\zeta_{\alpha\beta}(\mathbf{w}_\alpha - \mathbf{w}_\beta)\right] =$$
$$= {}^2/_5\, n_\alpha e_\alpha(\mathbf{r}_\alpha \times \mathbf{H}), \quad (2)$$
$$\sum_\alpha \rho_\alpha \mathbf{w}_\alpha = 0\,. \quad (3)$$

Here m_α, e_α, n_α, $\rho_\alpha = n_\alpha m_\alpha$ are, respectively, the mass, charge, density, and mass density of particles of type α; ρ is the mass density of the mixture; $\mathbf{u}$ is the mean mass velocity of the gas; and $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field strengths. Moreover,

$$\lambda_{\alpha\beta} \sim n_\alpha n_\beta \mu_{\alpha\beta}{}^{1/2} Q_{\alpha\beta}\,. \quad (4)$$

Here $\mu_{\alpha\beta}$ is the reduced mass and $Q_{\alpha\beta}$ is the mean effective collision cross section for particles of types α and β. It is convenient to express the quantity $\lambda_{\alpha\beta}$ in terms of the effective collision frequency $\tau_{\alpha\beta}{}^{-1}$

$$\lambda_{\alpha\beta} = n_\alpha \mu_{\alpha\beta}\tau_{\alpha\beta}{}^{-1} = n_\beta \mu_{\alpha\beta}\tau_{\beta\alpha}{}^{-1} = \lambda_{\beta\alpha}\,. \quad (5)$$

Expressions for $\tau_{\alpha\beta}^{-1}$ and the coefficients $b_{\alpha\beta}$, $b'_{\alpha\beta}$, $\zeta_{\alpha\beta}$ for different interaction laws are given in [4].

Equation (2) is solved for $\mathbf{r}_\gamma$. Setting the expressions thus obtained in (1), we arrive at a system of linear vector equations for the diffusion velocities $\mathbf{w}_\gamma$. These equations are linearly dependent, and so the actual number of equations necessary to determine the current density

$$\mathbf{j} = \sum_\gamma n_\gamma e_\gamma \mathbf{w}_\gamma \quad (6)$$

is less by one than the number of components. In the solution of the system (1)–(3) below, the equations for the electronic component (α = e) and N independent equations (1) for the neutral components ($\alpha = 1, \ldots, N$) are employed. Thanks to the conditions $m_e/m_\beta \ll 1$ and $b'_{e\beta} \sim (m_e/m_\beta) \ll 1$ for $\beta \neq e$ we may neglect in the equations for electrons, terms containing the heat fluxes of ions and neutrals. Omitting the last term on the right hand side of (1) for the same reason, and taking into consideration that $\mu_{e\beta} \approx m_e$, we have

$$\sum_{\beta\neq e}\tau_{e\beta}{}^{-1}[\mathbf{w}_e - \mathbf{w}_\beta + \zeta_{e\beta}\mathbf{r}_e] = -\frac{e}{m_e}\mathbf{E}' - \omega_e(\mathbf{w}_e\times\mathbf{k}), \quad (7)$$
$$\mathbf{r}_e + \omega_e\tau_e{}^*(\mathbf{r}_e\times\mathbf{k}) = -\frac{5}{2}\tau_e{}^*\sum_{\beta\neq e}\tau_{e\beta}{}^{-1}\zeta_{e\beta}(\mathbf{w}_e - \mathbf{w}_\beta). \quad (8)$$

Here

$$\omega_e = \frac{e}{m_e}H, \qquad e = |e_e|, \qquad \mathbf{k} = \frac{\mathbf{H}}{H}, \quad (9)$$
$$\frac{1}{\tau_e{}^*} = 0.4\tau_{ee}{}^{-1} + 2.5\sum_{\beta\neq e}\frac{1 - 0.48B_{e\beta}{}^*}{\tau_{e\beta}}. \quad (10)$$

In writing (10), the expressions for $b_{\alpha\beta}$, $b'_{\alpha\beta}$ given in [4] were employed. The coefficient $B_{\alpha\beta}{}^*$ depends only slightly on the character of electron scatter and differs little from unity. For the case of single ionization, $e_i = -e_e = e$ and $r_i = n_e$. Then

$$\mathbf{j} = n_e e(\mathbf{w}_i - \mathbf{w}_e)\,. \quad (11)$$

Taking into account that

$$\mathbf{w}_e = \sum_{\gamma\neq e}\frac{\rho_\gamma}{\rho}(\mathbf{w}_e - \mathbf{w}_\gamma) \quad (12)$$

in view of condition (3), and introducing the quantities

$$\mathbf{S}_\beta = n_e e(\mathbf{w}_i - \mathbf{w}_\beta) \quad (\beta = 1, \ldots, N), \qquad \mathbf{X}_e = n_e e\,\mathbf{r}_e, \quad (13)$$

we represent equations (7), (8) in the form

$$\mathbf{j} + \omega_e\tau_0(\mathbf{j}\times\mathbf{k}) = \quad (14)$$
$$= \sigma_0\mathbf{E}' + \tau_0\sum_{\beta=1}^N\frac{1}{\tau_{e\beta}}\mathbf{S}_\beta + \omega_e\tau_0\sum_{\beta=1}^N\frac{\rho_\beta}{\rho}(\mathbf{S}_\beta\times\mathbf{k}) + \nu_0\tau_0\mathbf{X}_e,$$
$$\mathbf{X}_e + \omega_e\tau_e{}^*(\mathbf{X}_e\times\mathbf{k}) = \frac{5}{2}\nu_0\tau_e{}^*\mathbf{j} - \frac{5}{2}\tau_e{}^*\sum_{\beta=1}^N\frac{\zeta_{e\beta}}{\tau_{e\beta}}\mathbf{S}_\beta. \quad (15)$$

Here

$$\sigma_0 = \frac{n_e e^2}{m_e}\tau_0, \qquad \frac{1}{\tau_0} = \sum_{\beta\neq e}\frac{1}{\tau_{e\beta}}, \qquad \nu_0 = \sum_{\beta\neq e}\frac{\zeta_{e\beta}}{\tau_{e\beta}}. \quad (16)$$

To determine S_β we have N independent equations (1) for $\alpha = 1, 2, \ldots, N$. Here, as in the equations for electrons, we neglect terms containing the heat fluxes of ions and neutrals. The latter procedure may be justified as follows. The expressions for $\mathbf{r}_i$ and $\mathbf{r}_\beta$ ($\beta = 1, 2, \ldots, N$), resulting from the solution of equations (2), contain terms proportional to S_β only, since $\mathbf{r}_e$ and $\mathbf{j}$ appear in them with coefficients $\sim m_e/m_\beta$. Moreover, the proportionality coefficients $\zeta_{i\beta}$ and $\zeta_{\alpha\beta}$ depend on ion-atom and atom-atom interactions only, and so for real interaction potentials $\zeta_{i\beta} \sim \zeta_{\alpha\beta} \leq 0.2$. Setting $\mathbf{r}_i$ and $\mathbf{r}_\beta$ in (1), we note that the additions to the coefficients for S_β turn out to be quadratic with respect to the quantities $\zeta_{i\beta}$ and $\zeta_{\alpha\beta}$, i.e., taking into account the heat fluxes of ions and neutrals introduces only insignificant corrections. (Estimates for a three-component plasma [3, 4] show that neglecting $\mathbf{r}_i$ and $\mathbf{r}_\alpha$ leads to an error not exceeding 2% in the final result.) Then, taking into account that $e_\alpha = 0$ for $\alpha = 1, 2, \ldots, N$, we may represent Eqs. (1) for the neutral components in the form

$$\sum_{\beta=1}^{N} a_{\alpha\beta} S_\beta = \lambda_{e\alpha}(\mathbf{j} - \zeta_{e\alpha}\mathbf{X}_e) + n_e e \frac{\rho_\alpha}{\rho}(\mathbf{j} \times \mathbf{H}), \quad (17)$$

$$a_{\alpha\alpha} = \sum_{\gamma \neq \alpha} \lambda_{\alpha\gamma} \quad (\gamma = e, i, 1, \ldots, N), \quad (18)$$

$$a_{\alpha\beta} = -\lambda_{\alpha\beta} \quad (\beta \neq \alpha).$$

Solving the equations, we have

$$S_\alpha = c_\alpha \mathbf{j} - d_\alpha \mathbf{X}_e + f_\alpha (\mathbf{j} \times \mathbf{k}), \quad (19)$$

$$c_\alpha = \sum_{\beta=1}^{N} \frac{|a|_{\beta\alpha}}{|a|}\lambda_{e\beta}, \quad d_\alpha = \sum_{\beta=1}^{N} \frac{|a|_{\beta\alpha}}{|a|}\zeta_{e\beta}\lambda_{e\beta}, \quad (20)$$

$$f_\alpha = n_e m_e \omega_e \sum_{\beta=1}^{N} \frac{\rho_\beta}{\rho}\frac{|a|_{\beta\alpha}}{|a|}.$$

Here $|a|$ is the determinant of the system and $|a|_{\beta\alpha}$ is the cofactor of the element $\beta\alpha$ of the determinant.

We note that the nondiagonal elements of the determinant depend on quantities which characterize atom-atom interactions only. The diagonal terms contain, together with other elements, the quantities $\lambda_{e\alpha}$ and $\lambda_{i\alpha}$. In this case, $\lambda_{e\alpha}/\lambda_{i\alpha} \sim m_e^{1/2}/\mu_{i\alpha}^{1/2}$ if $Q_{e\alpha} \sim Q_{i\alpha}$, and so $\lambda_{e\alpha} \ll \lambda_{i\alpha}$. Assuming that the cross sections $Q_{\alpha\beta}$ for atom-atom interactions have the same order of magnitude as $Q_{i\alpha}$, and estimating the coefficients in (19) under these conditions, we have

$$c_\alpha \lesssim (m_e/m_k)^{1/2}, \qquad d_\alpha < c_\alpha, \qquad f_\alpha \lesssim (m_e/m_k)^{1/2}\omega_e\tau_0.$$

Here the index k refers to the lightest neutral component in the mixture. Setting (19) in equations (14), (15) and solving them jointly, we arrive at the equation for $\mathbf{j}$, which when terms $\sim (m_e/m_k)^{1/2}$ are neglected assumes the form

$$A\mathbf{j} + B\omega_e\tau_0(\mathbf{j} \times \mathbf{k}) - C\omega_e^2\tau_0^2\,\mathbf{k}(\mathbf{jk}) = \sigma_0\mathbf{E}'. \quad (21)$$

Here

$$A = 1 - \frac{\Delta_0}{1 + \gamma^2\omega_e^2\tau_0^2} + \delta_0\omega_e^2\tau_0^2,$$

$$B = 1 + \gamma\frac{\Delta_0}{1 + \gamma^2\omega_e^2\tau_0^2},$$

$$C = \delta_0 + \gamma^2\frac{\Delta_0}{1 + \gamma^2\omega_e^2\tau_0^2}, \quad (22)$$

$$\Delta_0 = \frac{5}{2}\nu_0^2\tau_e^*\tau_0, \quad \gamma = \tau_e^*\tau_0^{-1},$$

$$\delta_0 = n_e m_e \tau_0^{-1}\sum_{\alpha,\beta}\frac{\rho_\alpha\rho_\beta}{\rho^2}\frac{|a|_{\beta\alpha}}{|a|}. \quad (23)$$

Equation (21) is the generalized Ohm's law for the multicomponent mixture under consideration. Solving it for $\mathbf{j}$, we have

$$\mathbf{j} = \sigma_\parallel \mathbf{E}_\parallel' + \sigma_\perp \mathbf{E}_\perp' + \sigma_H(\mathbf{k} \times \mathbf{E}'), \quad (24)$$

where $\mathbf{E}_\parallel' = \mathbf{k}(\mathbf{E}'\mathbf{k})$ and $\mathbf{E}_\perp' = \mathbf{k} \times (\mathbf{E}' \times \mathbf{k})$ are the components of $\mathbf{E}'$ corresponding to the parallel and perpendicular magnetic fields, and $\sigma_\parallel$, $\sigma_\perp$ and σ_H are the longitudinal, transverse and "Hall" conductivities given by the expressions

$$\sigma_\parallel = \frac{\sigma_0}{1 - \Delta_0}, \qquad \sigma_\perp = \sigma_0\frac{A}{A^2 + B^2\omega_e^2\tau_0^2}, \quad (25)$$

$$\sigma_H = \sigma_0\frac{B\omega_e\tau_0}{A^2 + B^2\omega_e^2\tau_0^2}.$$

The structure of the expressions for the electrical conductivities (25) remains the same in fact as in the case of a three-component plasma [3, 4]. The presence of several types of neutrals in the mixture leads only to an additional contribution to the coefficients τ_0^{-1}, $(\tau_e^*)^{-1}$ and ν_0, taking into account the interaction of the electrons with neutrals of each type.

The generalized expression for the coefficient δ_0 is also important. We note that it is precisely the term $\delta_0\omega_e^2\tau_0^2$ which describes the influence of the relative diffusion of heavy components (ions and neutrals) on the electrical conductivity across the magnetic field. If $\delta_0\omega_e^2\tau_0^2 \ll 1$, then the multicomponent plasma under consideration is described, with a good degree of approximation, by the model of a quasi-two-component medium [7]. The expressions for the electrical conductivity of such a plasma (25) then follow immediately from the solution of the equations for the electronic component (7) and (8), if we set $\mathbf{w}_\beta = 0$ in them, i.e., if we consider the velocities of the heavy components $\mathbf{u}_\beta$ approximately equal to the mean mass velocity of the gas $\mathbf{u}$. We now give the expressions for δ_0 for the cases when one and two types of neutrals are present in the mixture:

for a three-component plasma

$$\delta_0 = n_e m_e \tau_0^{-1}\left(\frac{\rho_1}{\rho}\right)^2\frac{1}{\lambda_{i1}} = \left(\frac{m_e}{\mu_{i1}}\right)^{1/2}\left(\frac{\rho_1}{\rho}\right)^2\frac{n_1Q_{e1} + n_iQ_{ei}}{n_1Q_{i1}}; \quad (26)$$

for a plasma with two types of neutrals

$$\delta_0 = \frac{n_e m_e \tau_0^{-1}}{\rho^2}\frac{\rho_1^2\lambda_{i2} + \rho_2^2\lambda_{i1} + (\rho_1 + \rho_2)^2\lambda_{12}}{\lambda_{i1}\lambda_{i2} + \lambda_{12}(\lambda_{i1} + \lambda_{i2})}. \quad (27)$$

If the relative concentration of charged particles in the mixture is not large ($n_i \ll n_1 + n_2$), the latter expression may be simplified:

$$\delta_0 = \frac{m_e^{1/2}(n_1Q_{e1} + n_2Q_{e2} + n_iQ_{ei})}{\mu_{i1}^{1/2}n_1Q_{i1} + \mu_{i2}^{1/2}n_2Q_{i2}}. \quad (28)$$

A typical example in which the results obtained above may find application, is the calculation of the electrical conductivity of mixtures with a lightly ionized additive, such as are employed in a series of magnetohydrodynamic devices. In estimating the influence of ion slip on the electrical conductivity of such mixtures the use of expressions (27) or (28) is more exact than expressions of the type (26).

We shall now estimate the role of "second-order" corrections to the theory. Clearly, their influence on the transverse and "Hall" conductivities notably decreases as the parameter $\omega_e^2\tau_0^2$ increases. Thus, if taking ion slip into account is important, i.e., $\delta_0\omega_e^2\tau_0^2 \sim 1$, then $\omega_e^2\tau_0^2 \gg 1$ since $\delta_0 \ll 1$. Under these conditions the contribution of "second-order" corrections turns out to be vanishingly small. For cases when $\omega_e\tau_0 \lesssim 1$, and also in calculating the longitudinal conductivity $\sigma_\parallel$, it is essential to take into account the corrections Δ_0. We shall examine the calculation of $\sigma_\parallel$ in greater detail.

In the limit of a weakly ionized gas when only the interactions of electrons with neutrals are important, we may represent the expression for the effective collision frequency τ_0^{-1} in the form

$$\tau_0^{-1} = \frac{4}{3}\bar{v}_e \sum_\beta n_\beta Q_{e\beta}, \qquad \bar{v}_e = \left(\frac{8kT}{\pi m_e}\right)^{1/2}, \tag{29}$$

$$Q_{e\beta} = \int x^5 \exp(-x^2)\, q_{e\beta}(v, \chi)(1-\cos\chi)\, d\Omega dx, \tag{30}$$

where $q_{e\beta}(v, \chi)$ is the effective differential cross section of elastic scattering of electrons by β-type neutrals, χ is the scattering angle, $d\Omega = \sin\chi\, d\chi d\varepsilon$ (ε is the azimuthal angle), $x^2 = (m_e/2kT)v^2$, and v is the electron velocity.

We have for "second-order" corrections

$$\Delta_0^{en} = \Big(\sum_\beta \zeta_{e\beta} n_\beta Q_{e\beta}\Big)^2 \Big/ \Big(\sum_\gamma n_\gamma Q_{e\gamma} \sum_\beta (1 - 0.48B_{e\beta}^*)\, n_\beta Q_{e\beta}\Big). \tag{31}$$

If $q(v, \chi) = \text{const}$, which corresponds to the model of interaction of hard elastic spheres, then $\zeta_{e\beta} = 0.2$, $B_{e\beta}^* = 1$ and $\Delta_0^{en} = 0.077$. In this case the expression for the conductivity $\sigma_\parallel$ assumes the form

$$\sigma_\parallel^{en} = 0.510\, \frac{e^2 n_e}{(m_e kT)^{1/2}} \Big(\sum_\beta n_\beta Q_{e\beta}\Big)^{-1}. \tag{32}$$

This result differs by only 4% from the exact Lorentz value $\sigma_\parallel$ of the interaction for this model (0.510 instead of 0.532 [6]). If the interaction of electrons with neutrals of different types is varied in character, then, generally speaking, Δ_0^{en} depends on the relative concentration of neutral components in the mixture. However, it must be noted that as a rule this correction is itself small. The contribution to Δ_0 in the other limiting case, when the gas is fully ionized, turns out to be more important. For this case $\zeta_{ei} = -0.6$, $B_{ei}^* = 1$ and $\Delta_0^{ei} = 0.482$. Using the expressions

$$\tau_{ei}^{-1} = \frac{4}{3} n_i \left(\frac{2\pi kT}{m_e}\right)^{1/2} \left(\frac{e^2}{kT}\right)^2 \ln\Lambda, \qquad \tau_{ee}^{-1} \approx \sqrt{2}\,\tau_{ei}^{-1}, \tag{33}$$

we have

$$\sigma_\parallel^{ei} = 0.582\, \frac{(kT)^{3/2}}{e^2 m_e^{1/2}} \frac{1}{\ln\Lambda}, \tag{34}$$

which practically coincides with the well-known result of Spitzer (0.582 instead of 0.591 [8]).

We shall now consider the case of an arbitrary degree of ionization of the mixture. Different interpolations are proposed in order to calculate the conductivity in this region, all of which are based on the assumption that the specific resistances due to electron-neutral and electron-ion collisions [9] are additive. If we leave aside the small differences in numerical coefficients for the limiting cases mentioned above, then the most often used expression for $\sigma_\parallel$ may be represented in the form

$$\frac{1}{\sigma_\parallel} = \frac{1}{\sigma_\parallel^{en}} + \frac{1}{\sigma_\parallel^{ei}}, \tag{35}$$

where $\sigma_\parallel^{en}$ and $\sigma_\parallel^{ei}$ are determined from expressions (32) and (34). Actually, as follows from (16), only the effective collision frequencies $\tau_{e\beta}^{-1}$ or specific resistances calculated only in first-approximation theory are additive. Hence a more accurate expression for $\sigma_\parallel$ is

$$\frac{1}{\sigma_\parallel} = \frac{1}{\sigma_\parallel^{en}} \frac{1-\Delta_0}{1-\Delta_0^{en}} + \frac{1}{\sigma_\parallel^{ei}} \frac{1-\Delta_0}{1-\Delta_0^{ei}}. \tag{36}$$

Comparison of (35) and (36) allows us to estimate the error involved in calculating the conductivity from the approximate formula (35). Both expressions give values of $\sigma_\parallel$ which coincide in the limiting cases but may differ noticeably for effective electron-neutral and electron-ion collision frequencies which are comparable in magnitude. As a rule, $\Delta_0 \ll 1$ in this region and the coefficient for the first term in (36) is of the order of two; thus, calculations from the approximate formula (35) may give values of $\sigma_\parallel$ 1.3–1.5 times higher than the more exact values of (36).

REFERENCES

1. T. G. Cowling, "The electrical conductivity of an ionized gas in a magnetic field with applications to the solar atmosphere and the ionosphere," Proc. Roy. Soc. A, vol. 183, p. 453, 1945.
2. H. Schirmer and T. Friedrich, "Elektrische Leitfähigkeit der Plasmen," Z. Physik, vol. 151, no. 174, pp. 375, 1958.
3. A. C. Pipkin, "Electrical conductivity of partially ionized gases," Phys. Fluids, vol. 4, p. 154, 1961.
4. V. M. Zhdanov, "Transport phenomena in a partially ionized gas," PMM, vol. 26, no. 2, 1962.
5. A. F. Nastoyashchii and L. D. Puzikov, "The equations of thermal and electrical conductivity in a partially ionized gas," PMTF, no. 5, 1962.
6. S. Chapman and T. Cowling, The Mathematical Theory of Nonuniform Gases [Russian translation], Izd. inostr. lit., 1960.
7. T. Kihara, "Thermodynamical foundations of plasma theory," J. Phys. Soc. Japan, vol. 14 (2), p. 128, 1959.
8. L. Spitzer, Physics of Fully Ionized Gases [Russian translation], Izd. inostr. lit., 1957.
9. S. C. Lin, E. L. Resler, and A. R. Kantorowitz, "Electrical conductivity of highly ionized argon produced by shock waves," J. Appl. Phys., vol. 26, p. 95, 1955.

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